

ON THE FINITE TIME BLOWUP FOR MASS-CRITICAL HARTREE EQUATIONS

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ABSTRACT. We consider the fractional Schrödinger equations with focusing Hartree type nonlinearity. When the energy is negative, we use a Glassey's virial type argument to show the finite time blow-up of solutions.

1. INTRODUCTION

In this paper we consider the Cauchy problem of the focusing fractional nonlinear Schrödinger equations:

$$(1.1) \quad \begin{cases} i\partial_t u = |\nabla|^\alpha u + F(u), & \text{in } \mathbb{R}^{1+n} \times \mathbb{R}, \\ u(x, 0) = \varphi(x) & \text{in } \mathbb{R}^n, \end{cases}$$

where $|\nabla| = (-\Delta)^{\frac{1}{2}}$, $n \geq 2$, $\alpha \geq 1$. $F(u)$ is a nonlocal nonlinear term of Hartree type as follows:

$$F(u)(x) = -\left(\frac{\psi(\cdot)}{|\cdot|^\gamma} * |u|^2\right)(x) u(x) \equiv -V_\gamma(|u|^2)(x) u(x),$$

where $0 \leq \psi \in L^\infty(\mathbb{R}^n)$, and $0 < \gamma < n$. We refer (1.1) is focusing as $-V_\gamma(|u|^2)$ plays a role of attractive self-reinforcing potential. We also use a simpler notation V_γ to denote $V_\gamma(|u|^2)$.

In case when ψ is homogeneous of degree zero (e.g. $\psi \equiv 1$), the equation (1.1) has scaling invariance. That is, if u is a solution of (1.1), then u_λ for $\lambda > 0$, given by

$$u_\lambda(t, x) = \lambda^{-\frac{\gamma-\alpha}{2} + \frac{n}{2}} u(\lambda^\alpha t, \lambda x),$$

is also a solution. \dot{H}^{sc} -norm of solution is preserved under scaling at $s_c = \frac{\gamma-\alpha}{2}$.

The solution u of (1.1) formally satisfies the mass and energy conservation laws:

$$(1.2) \quad \begin{aligned} m(u) &= \|u(t)\|_{L^2}^2, \\ E(u) &= K(u) + V(u), \end{aligned}$$

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where

$$K(u) = \frac{1}{2} \langle u, |\nabla|^\alpha u \rangle, \quad V(u) = -\frac{1}{4} \langle u, V_\gamma(|u|^2)u \rangle.$$

Here $\langle \cdot, \cdot \rangle$ is the complex inner product in L^2 . In view of the scaling invariance and conservation laws - when each conserved quantity is invariant under scaling - we say the equation (1.1) is mass-critical if $\gamma = \alpha$ and energy-critical if $\gamma = \alpha/2$.

The purpose of this paper is to show the finite time blow-up of solutions to the fractional or high order equations when (1.1) is mass-critical. For the usual Schrödinger equations ($\alpha = 2$), Glassey [7] introduced a convexity argument to show the existence of finite time blow-up solutions. Indeed, if $\psi = 1$, $2 \leq \gamma < \min(n, 4)$, $n \geq 3$ and $\varphi \in H^{\frac{\gamma}{2}}(\mathbb{R}^n)$ with $x\varphi \in L^2$, then

$$\|xu(t)\|_{L^2}^2 \leq 8t^2 E(\varphi) + 4t \langle \varphi, A\varphi \rangle + \|x\varphi\|_{L^2}^2,$$

where A is the dilation operator $\frac{1}{2i}(\nabla \cdot x + x \cdot \nabla)$. This implies that if $E(\varphi) < 0$, then the maximal time of existence $T^* < \infty$. For details, see [1] and Section 3 below. The focusing nonlinearity serves as an attracting potential. If the energy is negative (i.e. the magnitude of the potential energy $V(u)$ is larger than that of kinetic part $K(u)$), then self-attracting power overwhelms the dynamics and so it results in a collapse in a finite time.

In the fractional or high order equations, the analogue to the second moment is the quantity

$$\mathcal{M}(u) \equiv \langle u, x \cdot |\nabla|^{2-\alpha} xu \rangle.$$

This variant was first utilized by Fröhlich and Lenzmann [6], when they studied the same problem for the semirelativistic nonlinear Schrödinger equations ($\alpha = 1$). More precisely, they have shown

$$\mathcal{M}(u(t)) \leq 2t^2 E(\varphi) + 2t (\langle \varphi, A\varphi \rangle + C\|\varphi\|_{L^2}^4) + \mathcal{M}(\varphi)$$

for $\psi = e^{-\mu|x|}$ ($\mu \geq 0$), $\gamma = 1$, $\varphi \in H_{rad}^2(\mathbb{R}^3)$ with $|x|^2\varphi \in L^2$. Here the space X_{rad} denotes the subset of the space X which consists of radial functions. The quartic term $\|\varphi\|_{L^2}^4$ appears due to the commutator $[|x|^2 V_\gamma, |\nabla|]$ which was controlled by utilizing Newton's theorem in \mathbb{R}^3 .

Compared to the usual case ($\alpha = 2$), when $\alpha \neq 2$, the presence of $|\nabla|^{2-\alpha}$ gives rise to certain singular integrals which necessitate the use of commutator estimates. In order to handle different orders, the main issue is to find the estimate for the commutator $[|x|^2 V_\gamma, |\nabla|^{2-\alpha}]$ since the Newton's theorem is generally not available except for $\alpha = 1$ in \mathbb{R}^3 . Previously, when $1 < \alpha < 2$, in [2] the authors considered a short-range truncation function ψ given by

$$(1.3) \quad \psi(x) = \chi_{\{|x| \geq r_0\}}(x) \quad \text{for some } 0 < r_0 \ll 1$$

in some mass-supercritical regime ($\gamma > \alpha + 1$) and they showed that

$$(1.4) \quad \mathcal{M}(u(t)) \leq 2\alpha^2 t^2 E(\varphi) + 2\alpha t \left(\langle \varphi, A\varphi \rangle + Cr_0^{-(\gamma-\alpha)} \|\varphi\|_{L^2}^4 \right) + \mathcal{M}(\varphi).$$

In the argument of [2], the truncation is crucial and their result seems to suggest that the blow-up time may be related to the size of truncation.

Main improvement here is to remove the truncation and so extends the commutator estimate to general cases $[|x|^2 V_\gamma, |\nabla|^{2-\alpha}]$. For this purpose we use Stein-Weiss inequality (2.12), combined with a convolution estimate in Lemma 2.3. To close our argument, we also need to obtain estimate on the propagation of the moments $\|xu\|_{L^2}$ and $\|x|\nabla u\|_{L^2}$. It is done under some regularity assumption (at least H^α regularity)¹ which is required to get estimates for the commutators $[|\nabla|^\alpha, |x|^2]$ and $[V_\gamma, \nabla \cdot (x \cdot |\nabla|^{2-\alpha} x \nabla)]$.

The Hartree nonlinearity is essentially a cubic one, though it is convolved with the potential. Thus, by fairly a standard argument one can show the local well-posedness of the Cauchy problem for suitably regular initial data. Indeed, we have the following local well-posedness: if $s \geq \frac{\gamma}{2}$, a unique solution $u \in C([0, T^*]; H^s) \cap C^1([0, T^*]; H^{s-\alpha})$ exists within the maximal existence time interval $[0, T^*)$, and that if $T^* < \infty$, then $\lim_{t \nearrow T^*} \|u(t)\|_{H^{\frac{\gamma}{2}}} = \infty$. We append the local well-posedness for general case of $\alpha > 0$ in the last section.

To state our main theorem, let us define a Sobolev index α^* by $\alpha^* = \max(\alpha, \frac{\gamma+1}{2})$ if α is even integer and $\alpha^* = \max(2k+1, \frac{\gamma+1}{2})$ if α is not even where k is the least integer such that $k > \frac{\alpha}{2}$. We state theorems in two cases, separately, for $1 < \alpha < 2$ and $2 < \alpha < n-1$.

Theorem 1.1. *Let ψ be a nonnegative, radial bounded function with $\psi' \leq 0$ and $|\psi'(\rho)| \leq C\rho^{-1}$ for some positive constant C . Let $1 < \alpha < 2$, $\gamma = \alpha$ and $n \geq 4$. Then for any $\varphi \in H_{rad}^{\alpha^*}$ with $x\varphi, |x|\nabla\varphi \in L^2$ and $E(\varphi) < 0$, the maximal time of existence T^* of the solution u to (1.1) is finite.*

On the other hand, when $\alpha > 2$, we have the following.

Theorem 1.2. *Let ψ be a nonnegative, radial bounded function with $\psi' \leq 0$. Let $\gamma = \alpha$, $2 < \alpha < 1 + \frac{n}{2}$ and $n \geq 4$. Then for any $\varphi \in H_{rad}^{\alpha^*}$ and $x\varphi, |x|\nabla\varphi \in L^2$ with $E(\varphi) < 0$, the maximal time of existence T^* of the solution u to (1.1) is finite.*

The restriction $n \geq 4$ results from the use of Stein-Weiss inequality and Lemma 2.3 for which the condition $\alpha < \min(1 + \frac{n}{2}, n-1)$ is necessary. For the proof of

¹ Such assumption is not necessary for the usual Schrödinger equation.

theorems we show that the mean dilation is decreasing if $E(\varphi) < 0$.

$$(1.5) \quad \frac{d}{dt} \langle u, Au \rangle \leq 2\alpha E(\varphi)$$

This holds true in general whenever $\gamma \geq \alpha$ and $\psi' \leq 0$. Then if $\gamma = \alpha$, from the estimate of the change of $\mathcal{M}(u)$ in time, then for any $t \in [0, T^*)$, we obtain

$$(1.6) \quad \mathcal{M}(u) \leq 2\alpha^2 t^2 E(\varphi) + 2\alpha t (\langle \varphi, A\varphi \rangle + C \|\varphi\|_{L^2}^4) + \mathcal{M}(\varphi).$$

To get a validity of (1.5) and (1.6), we need to estimate the evolution of moments $\|xu\|_{L^2}$ and $\|x|\nabla u\|_{L^2}$ on the time interval $[0, T^*)$.

If one considers truncated potential as in [2], then the range $\gamma > \alpha + 1$ for (1.4) improves to $\gamma > \alpha$ by the argument of this paper. We leave the details to the readers. We finally remark on the power type nonlinearity. Since our argument is based on the regularity (H^{α^*}) assumption and Stein-Weiss inequality, a different approach seems to be needed to control the commutators.

The rest of paper is organized as follows: In Section 2 we show finite time blowup under the regularity and moment assumption. In Section 3 we estimate the propagation of moment. The last section is devoted to the local well-posedness.

Notations. We will use the notations $|\nabla| = \sqrt{-\Delta}$, $\dot{H}_r^s = |\nabla|^{-s} L^r$, $\dot{H}^s = \dot{H}_2^s$, and $H_r^s = (1 - \Delta)^{-s/2} L^r$, $H^s = H_2^s$. The space X_{rad} denotes the subset of the space X which consists of radial functions. $A \lesssim B$ and $A \gtrsim B$ means that $A \leq CB$ and $A \geq C^{-1}B$, respectively for some $C > 0$. As usual different positive constants possibly depending on n, α and γ are denoted by the same letter C .

2. FINITE TIME BLOW-UP

In this section we consider the finite time blow-up of solutions to the Cauchy problem (1.1) of mass-critical potentials. We begin with the dilation operator A . With more general assumption for ψ and γ we obtain an estimate for the time evolution of average of A .

Lemma 2.1. *Let ψ be radially symmetric function such that ψ' exists a.e. and $\psi' \leq 0$ a.e. Suppose that $u \in H^{\alpha^*}$ and $xu(t), |x|\nabla u(t) \in L^2$ for $t \in [0, T^*)$, where T^* is the maximal existence time. Then for $\gamma \geq \alpha$*

$$(2.1) \quad \frac{d}{dt} \langle u, Au \rangle \leq 2\alpha E(\varphi).$$

Proof of Lemma 2.1. Since $u \in H^{\alpha^*}$ and $|x|u, x \cdot \nabla u \in L^2$, $\langle u, Au \rangle$ is well-defined and so is

$$(2.2) \quad \frac{d}{dt} \langle u, Au \rangle = i \langle u, [H, A]u \rangle,$$

where $H = |\nabla|^\alpha - V_\gamma$. Here $[H, A]$ denotes the commutator $HA - AH$. Using the identity $[|\nabla|^\alpha, x] = -\alpha|\nabla|^{\alpha-2}\nabla$, we have

$$(2.3) \quad [|\nabla|^\alpha, A] = -i\alpha|\nabla|^\alpha.$$

Similarly,

$$(2.4) \quad [-V_\gamma, A] = -i(x \cdot \nabla)V_\gamma.$$

Substituting (2.3) and (2.4) into (2.2), we get

$$(2.5) \quad \frac{d}{dt}\langle u, Au \rangle = \alpha\langle u, |\nabla|^\alpha u \rangle + \langle u, (x \cdot \nabla)V_\gamma u \rangle.$$

For Hartree type V_γ , we have

$$\begin{aligned} (x \cdot \nabla)V_\gamma &= -\gamma \int \frac{\psi(|x-y|)}{|x-y|^\gamma} |u(y)|^2 dy + \int \frac{\psi'(|x-y|)}{|x-y|^\gamma} |x-y| |u(y)|^2 dy \\ &\quad - \int \left(\gamma \frac{\psi(|x-y|)}{|x-y|^{\gamma+1}} - \frac{\psi'(|x-y|)}{|x-y|^\gamma} \right) \frac{y \cdot (x-y)}{|x-y|} |u(y)|^2 dy, \\ \langle u, (x \cdot \nabla)V_\gamma u \rangle &= 4\gamma V(u) + \iint \frac{|x-y|\psi'(|x-y|)}{|x-y|^\gamma} |u(x)|^2 |u(y)|^2 dx dy \\ &\quad - \langle u, (x \cdot \nabla)V_\gamma u \rangle, \end{aligned}$$

which implies

$$\langle u, (x \cdot \nabla)V_\gamma u \rangle = 2\gamma V(u) + \frac{1}{2} \iint \frac{|x-y|\psi'(|x-y|)}{|x-y|^\gamma} |u(x)|^2 |u(y)|^2 dx dy.$$

Substituting this into (2.5) gives

$$\begin{aligned} \frac{d}{dt}\langle u, Au \rangle &\leq 2\alpha E(\varphi) + 2(\gamma - \alpha)V(u) \\ &\quad + \frac{1}{2} \iint (|x-y|\psi'(|x-y|)) \frac{|u(x)|^2 |u(y)|^2}{|x-y|^\alpha} dx dy. \end{aligned}$$

Since $\gamma \geq \alpha$ and $\psi'(|x|) \leq 0$, we get (2.1). This completes the proof of Lemma 2.1. \square

Next we consider the nonnegative quantity $\mathcal{M}(u) = \langle u, Mu \rangle$ with the virial operator

$$M \equiv x \cdot |\nabla|^{2-\alpha} x = \sum_{k=1}^n x_k |\nabla|^{2-\alpha} x_k.$$

From the regularity and decay condition on u the quantity $\mathcal{M}(u)$ is well-defined and finite for all $t \in [0, T^*)$ and so is

$$(2.6) \quad \frac{d}{dt}\mathcal{M}(u) = i\langle u, [H, M]u \rangle = i\langle u, [|\nabla|^\alpha, M]u \rangle - i\langle u, [V_\gamma, M]u \rangle.$$

Lemma 2.2. *Under the same condition as in Theorems 1.1 and 1.2, we have*

$$(2.7) \quad \frac{d}{dt} \mathcal{M}(u) \leq 2\alpha \langle u, Au \rangle + C \|\varphi\|_{L^2}^4,$$

where C is a positive constant depending only on n, α, γ but not on u, φ .

Theorems 1.1 and 1.2 follow immediately from Lemma 2.1 and 2.2.

Proof of Lemma 2.2. Using the identity $|\nabla|^\alpha x = x|\nabla|^\alpha - \alpha|\nabla|^{\alpha-2}\nabla$, we first have the identity:

$$[|\nabla|^\alpha, M] = |\nabla|^\alpha x \cdot |\nabla|^{2-\alpha} x - x \cdot |\nabla|^{2-\alpha} x |\nabla|^\alpha = -\alpha(x \cdot \nabla + \nabla \cdot x).$$

For a smooth function v we get

$$\begin{aligned} [v, M] &= vx \cdot |\nabla|^{2-\alpha} x - x \cdot |\nabla|^{2-\alpha} xv \\ &= v|x|^2|\nabla|^{2-\alpha} - (2-\alpha)vx \cdot \nabla|\nabla|^{-\alpha} - |\nabla|^{2-\alpha}|x|^2v - (2-\alpha)|\nabla|^{-\alpha}\nabla \cdot xv \\ &= [|x|^2v, |\nabla|^{2-\alpha}] + (\alpha-2) \left(vx \cdot \frac{\nabla}{|\nabla|} |\nabla| |\nabla|^{-\alpha} + |\nabla| |\nabla|^{-\alpha} \frac{\nabla}{|\nabla|} \cdot xv \right). \end{aligned}$$

We first consider the high order case $\alpha > 2$. By density argument we may assume that $v = V_\gamma$ in the above identity. Thus it suffices to show that

$$(2.8) \quad \begin{aligned} &|\langle u, [|x|^2 V_\gamma, |\nabla|^{2-\alpha}] u \rangle| + |\langle u, \left(V_\gamma x \cdot \frac{\nabla}{|\nabla|} |\nabla| |\nabla|^{-\alpha} + |\nabla| |\nabla|^{-\alpha} \frac{\nabla}{|\nabla|} \cdot x V_\gamma \right) u \rangle| \\ &\lesssim \|\varphi\|_{L^2}^4. \end{aligned}$$

The first term of LHS of (2.8) is rewritten as

$$(2.9) \quad 2|\operatorname{Im} \langle u, |x|^2 V_\gamma |\nabla|^{2-\alpha} u \rangle|.$$

To handle this we recall the following weighted convolution estimate (see [5, 3]):

Lemma 2.3. *Let $0 < \gamma < n-1$ and $n \geq 2$, Then for any $f \in L_{rad}^1$ and $x \neq 0$*

$$(2.10) \quad \int |x-y|^{-\gamma} |f(y)| dy \lesssim |x|^{-\gamma} \|f\|_{L^1}.$$

From Lemma 2.3 and mass conservation (2.9) is bounded by

$$(2.11) \quad C \|\psi\|_{L^\infty} \|\varphi\|_{L^2}^2 \int |u(x)| |x|^{-(\alpha-2)} \int |x-y|^{-(n-\alpha+2)} |u(y)| dy dx.$$

To estimate this, we make use of the Stein-Weiss inequality [11]: for $f \in L^p$ with $1 < p < \infty$, $0 < \lambda < n$, $\beta < \frac{n}{p}$, and $n = \lambda + \beta$

$$(2.12) \quad \| |x|^{-\beta} (|\cdot|^{-\lambda} * f) \|_{L^p} \lesssim \|f\|_{L^p}.$$

Applying (2.12) with $p = 2$, $\beta = \alpha - 2$ and $\lambda = n - (\alpha - 2)$, (2.11) is bounded by $C \|\varphi\|_{L^2}^4$.

We write the second term of LHS of (2.8) as

$$2|\operatorname{Im}\langle u, \left(V_\gamma x \cdot \frac{\nabla}{|\nabla|} |\nabla| |\nabla|^{-\alpha} + |\nabla| |\nabla|^{-\alpha} \frac{\nabla}{|\nabla|} \cdot xv \right) u \rangle|,$$

which is bounded from Lemma 2.3 by

$$C\|\psi\|_{L^\infty}\|\varphi\|_{L^2}^2 \int |u(x)||x|^{-(\alpha-1)} \int |x-y|^{-(n-(\alpha-1))} \left| \left(\frac{\nabla}{|\nabla|} u \right)(y) \right| dy.$$

Applying (2.12) with $p = 2$, $\beta = \alpha - 1$ and $\lambda = n - (\alpha - 1)$ and then by Plancherel's theorem we get the desired bound (2.8).

Now we consider the fractional case $1 < \alpha < 2$. The first term of LHS of (2.8) equals

$$\sum_{j=1}^n \langle u, [T_j, g] \partial_j u + T_j(\partial_j g u) \rangle,$$

where $T_j = -|\nabla|^{2-\alpha}(-\Delta)^{-1}\partial_j$ and $g = |x|^2 V_\gamma$. By similar argument in the proof of Corollary of p.309, [10] one can see that

$$(2.13) \quad \|[T_j, g] \partial_j\|_{L^2 \rightarrow L^2} \lesssim \|g\|_{\dot{\Lambda}^{2-\alpha}},$$

where $\|g\|_{\dot{\Lambda}^{2-\alpha}} = \sup_{x \neq y \in \mathbb{R}^n} \frac{|g(x) - g(y)|}{|x - y|^{2-\alpha}}$. For details see [2]. If $x \neq y$, then

$$|g(x) - g(y)| \leq |x - y| \int_0^1 |\nabla g(z_s)| ds, \quad z_s = x + s(y - x).$$

Since $|\psi'(\rho)| \leq C\rho^{-1}$ for $\rho > 0$, from Lemma 2.3 and mass conservation it follows that

$$|\nabla g(z_s)| \lesssim |z_s|^{1-\alpha} \|u\|_{L^2}^2 = \|x - s|x - y|\|^{1-\alpha} \|\varphi\|_{L^2}^2,$$

provided $\alpha < n - 2$. By a simple calculation we see that if $0 < \theta < 1$, then

$$\sup_{a>0} \int_0^1 |a - s|^{-\theta} ds \leq C_\theta.$$

Thus from this we get that

$$|g(x) - g(y)| \lesssim |x - y|^{2-\alpha} \|\varphi\|_{L^2}^2,$$

which implies that

$$(2.14) \quad \|[T_j, g] \partial_j\|_{L^2 \rightarrow L^2} \lesssim \|\varphi\|_{L^2}.$$

On the other hand, since the kernel $k_j(x)$ of T_j is bounded by $C|x|^{-(n-\alpha+1)}$, from the duality and Lemma 2.3 with $\alpha < n - 2$

$$\begin{aligned} |\langle u, T_j((\partial_j g)u) \rangle| &= |\langle T_j^* u, (\partial_j g)u \rangle| \\ &\lesssim \|u\|_{L^2} \| |\partial_j g(\cdot)| \int |\cdot - y|^{-(n-\alpha+1)} |u(y)| dy \|_{L^2} \\ &\lesssim \|u\|_{L^2}^3 \| |\cdot|^{1-\alpha} \int |\cdot - y|^{-(n-\alpha+1)} |u(y)| dy \|_{L^2}, \end{aligned}$$

where T_j^* is the dual operator of T_j . Using the Stein-Weiss inequality (2.12) for $\beta = \alpha - 1$, $\lambda = n - \alpha + 1$ and $p = 2$, we get

$$(2.15) \quad |\langle u, T_j, \partial_j g u \rangle| \lesssim \|\varphi\|_{L^2}^4.$$

The second term of LHS of (2.8) can be treated by the same way as the high order case and is bounded by $C\|\varphi\|_{L^2}^4$. This together with (2.14) and (2.15) gives the desired bound (2.8). \square

3. PROPAGATION OF THE MOMENT

We now show estimates on the propagation of the moments $\|xu\|_{L^2}$ and $\|x|\nabla u\|_{L^2}$. The fractional case $1 < \alpha < 2$ was shown in [2]. So, in this section we consider only high order case $\alpha > 2$. Before getting started, let us denote the kernels of Bessel potential $D^{-\beta}$ and $|\nabla|^\alpha D^{-2k}$ ($\beta = \alpha - 2k, 2k > \alpha$) by $G_\beta(x)$ and $K(x)$, respectively. Then

$$K(x) = \sum_{j=0}^{\infty} A_j G_{2j+\beta}(x),$$

where the coefficients A_j are given by the expansion $(1-t)^{\frac{\alpha}{2}} = \sum_{j=0}^{\infty} A_j t^j$ for $|t| < 1$ with $\sum_{j \geq 0} |A_j| < \infty$. One can show that $(1+|x|)^\ell K \in L^1$ for $\ell \geq 1$ and has decreasing radial and integrable majorant. In fact, from the integral representation

$$G_{2j+\beta}(x) = \frac{1}{(4\pi)^{n/2} \Gamma(j + \beta/2)} \int_0^\infty \lambda^{(2j+\beta-n)/2-1} e^{-|x|^2/4\lambda} e^{-\lambda} d\lambda,$$

we have for j with $2j + \beta < n$

$$(3.1) \quad G_{2j+\beta}(x) \leq C(|x|^{-n+2j+\beta} \chi_{\{|x| \leq 1\}}(x) + e^{-c|x|} \chi_{\{|x| > 1\}}(x)),$$

and for j with $2j + \beta \geq n$

$$(3.2) \quad G_{2j+\beta}(x) \leq C(\chi_{\{|x| \leq 1\}}(x) + e^{-c|x|} \chi_{\{|x| > 1\}}(x)).$$

Here the constants c and C of (3.1) and (3.2) are independent of j . So, the function $(1+|x|)^\ell G_{2j+\beta}$ has a decreasing radial and integrable majorant, which is chosen uniformly on j , and so does K . For details see p.132–135 of [9].

Proposition 3.1. *Let T^* be the maximal existence time of solution $u \in C([0, T^*]; H^{\alpha*})$ to (1.1). If $x\varphi, |x|\nabla\varphi \in L^2$, then $xu(t), |x|\nabla u(t) \in L^2$ for all $t \in [0, T^*)$. Moreover, we have for $t \in [0, T^*)$*

$$(3.3) \quad \|xu(t)\|_{L^2} \leq \|x\varphi\|_{L^2} + C \int_0^t \|u(t')\|_{H^{\alpha*-1}} dt',$$

$$(3.4) \quad \||x|\nabla u(t)\|_{L^2} \leq \||x|\nabla\varphi\|_{L^2} + C \int_0^t \|u\|_{H^{\alpha*}} (1 + \|u\|_{H^{\alpha*}} \|xu\|_{L^2}) dt'.$$

Proof of Proposition 3.1. We first consider the case $\alpha = 2k$ for $k \geq 2$. Let us set $\psi_\varepsilon(x) = e^{-\varepsilon|x|^2}$ and

$$\mathbf{m}_\varepsilon(t) = \langle u(t), |x|^2 \psi_\varepsilon^2 u(t) \rangle.$$

From the regularity of u it follows that

$$\begin{aligned} \mathbf{m}'_\varepsilon(t) &= i \langle u, [|\nabla|^\alpha, |x|^2 \psi_\varepsilon^2] u \rangle \\ (3.5) \quad &= i \langle x \psi_\varepsilon u, [|\nabla|^\alpha, x \psi_\varepsilon] u \rangle + i \langle u, [|\nabla|^\alpha, x \psi_\varepsilon] \cdot x \psi_\varepsilon u \rangle \\ &= -2 \operatorname{Im} \langle x \psi_\varepsilon u, [|\nabla|^\alpha, x \psi_\varepsilon] u \rangle. \end{aligned}$$

Since $\alpha = 2k$, by integration by parts we have

$$\begin{aligned} \mathbf{m}'_\varepsilon(t) &= -2 \operatorname{Im} \langle x \psi_\varepsilon u, [|\nabla|^\alpha, x \psi_\varepsilon] u \rangle \\ &= -2 \operatorname{Im} \langle x \psi_\varepsilon u, -2k \nabla (-\Delta)^{k-1} (\psi_\varepsilon u) + x (-\Delta)^k (\psi_\varepsilon u) - x \psi_\varepsilon (-\Delta)^k u \rangle. \end{aligned}$$

Since the term $x(-\Delta)^k(\psi_\varepsilon u) - x\psi_\varepsilon(-\Delta)^k u$ consists of at least one derivative of ψ_ε and $\sum_{1 \leq |\ell| \leq 2k} \sup_x (1 + |x|) |\partial^\ell \psi_\varepsilon(x)| < \infty$ uniformly on ε , Cauchy-Schwarz inequality gives

$$\mathbf{m}'_\varepsilon(t) \leq C \|u(t)\|_{H^{\alpha-1}} \sqrt{\mathbf{m}_\varepsilon(t)},$$

where C is independent of ε . Then by a simple manipulation and integration we get for all $t \in [0, T^*)$

$$\sqrt{\mathbf{m}_\varepsilon(t)} \leq \sqrt{\mathbf{m}_\varepsilon(0)} + C \int_0^t \|u(t')\|_{H^{\alpha-1}} dt'.$$

Letting $\varepsilon \rightarrow 0$, we get the desired inequality (3.3).

Now let us set $v = \partial_j u$. Then one can easily see that

$$(3.6) \quad \frac{d}{dt} \langle v, |x|^2 \psi_\varepsilon^2 v \rangle = i \langle v, [|\nabla|^\alpha, |x|^2 \psi_\varepsilon^2] v \rangle + 2 \operatorname{Im} \langle x \psi_\varepsilon [\partial_j, V_\gamma] u, x \psi_\varepsilon \partial_j u \rangle.$$

By Hardy-Sobolev inequality it follows that

$$\|\partial_j V_\gamma\|_{L^\infty} \lesssim \sup_x \int |x-y|^{-\gamma-1} |u(y)|^2 dy \lesssim \|u\|_{H^{\frac{\gamma+1}{2}}}^2.$$

Thus we have

$$|\langle x \psi_\varepsilon [\partial_j, V_\gamma] u, x \psi_\varepsilon v \rangle| = |\langle x \psi_\varepsilon (\partial_j V_\gamma) u, x \psi_\varepsilon v \rangle| \lesssim \|u\|_{H^{\alpha*}}^2 \|xu\|_{L^2} \|x \psi_\varepsilon v\|_{L^2}.$$

The first term of the RHS of (3.6) can be estimated as above. So, we get

$$(3.7) \quad \| |x| \nabla u \|_{L^2}^2 \leq \| |x| \nabla \varphi \|_{L^2}^2 + C \int_0^t \|u\|_{H^{\alpha*}} (1 + \|u\|_{H^{\alpha*}} \|xu\|_{L^2}) \| |x| \nabla u \|_{L^2} dt'$$

for all $t \in [0, T^*)$. Gronwall's inequality gives (3.4).

Now we consider the case of non-even α . Let k be the least integer such that $2k > \alpha$ and set $\beta = 2k - \alpha$. By differentiating in time we have

$$\begin{aligned} \mathbf{m}'_\varepsilon(t) &= i\langle u, [|\nabla|^\alpha, |x|^2\psi_\varepsilon^2]u \rangle = -2\text{Im}\langle x\psi_\varepsilon u, [|\nabla|^\alpha, x\psi_\varepsilon]u \rangle \\ &= -2\text{Im}\langle x\psi_\varepsilon u, [|\nabla|^\alpha D^{-2k}, x\psi_\varepsilon]D^{2k}u \rangle \\ &\quad - 2\text{Im}\langle |\nabla|^\alpha D^{-2k}(x\psi_\varepsilon u), [D^{2k}, x\psi_\varepsilon]u \rangle \\ &\equiv I + II, \end{aligned}$$

where $D = \sqrt{1 - \Delta}$.

Then we first estimate I as follows:

$$\begin{aligned} I &= -2\text{Im}\left\langle x\psi_\varepsilon u, \int K(x-y)(y\psi_\varepsilon(y) - x\psi_\varepsilon(x))D^{2k}u(y) dy \right\rangle \\ &\lesssim \|x|K\|_{L^1}\|u\|_{H^{2k}}\sqrt{\mathbf{m}_\varepsilon}. \end{aligned}$$

To treat II we set $v = |\nabla|^\alpha D^{-2k}x\psi_\varepsilon u$. Then since $\sum_{1 \leq |\ell| \leq 2k} \sup_x (1+|x|)|\partial^\ell \psi_\varepsilon(x)| < \infty$ uniformly on ε , by Cauchy-Schwarz inequality and Plancherel's theorem, we get

$$\begin{aligned} II &= -2\text{Im} \sum_{l=1}^k \binom{k}{l} \langle v, [(-\Delta)^l, x\psi_\varepsilon]u \rangle \\ &= -2\text{Im} \sum_{l=1}^k \binom{k}{l} \langle v, -2l\nabla(-\Delta)^{l-1}(\psi_\varepsilon u) + x(-\Delta)^l(\psi_\varepsilon u) - x\psi_\varepsilon(-\Delta)^l u \rangle \\ &\lesssim \|u\|_{H^{2k-1}}\sqrt{\mathbf{m}_\varepsilon}. \end{aligned}$$

Therefore, by combining the estimates for I and II and repeating the same argument as before, we have

$$\sqrt{\mathbf{m}_\varepsilon(t)} \leq \sqrt{\mathbf{m}_\varepsilon(0)} + C \int_0^t \|u(t')\|_{H^{2k}} dt',$$

which gives

$$(3.8) \quad \|xu\|_{L^2} \leq \|x\varphi\|_{L^2} + C \int_0^t \|u(t')\|_{H^{2k}} dt' \quad \text{for all } t \in [0, T^*].$$

Now by using the same estimates as (3.6) and (3.7) together with (3.8), we get

$$\|x|\nabla u\|_{L^2} \leq \|x|\nabla\varphi\|_{L^2} + C \int_0^t \|u(t')\|_{H^{2k+1}} (1 + \|u(t')\|_{H^{2k+1}} \|xu(t')\|_{L^2}) dt'$$

for all $t \in [0, T^*)$, which means (3.4). This completes the proof of Proposition 3.1. \square

4. APPENDIX

In this section we show the local well-posedness of Hartree equation (1.1). Here we only assume that $\alpha > 0$ and $\psi \in L^\infty$.

Proposition 4.1. *Let $\psi \in L^\infty$. Let $\alpha > 0$, $0 < \gamma < n$ and $n \geq 1$. Suppose $\varphi \in H^s(\mathbb{R}^n)$ with $s \geq \frac{\gamma}{2}$. Then there exists a positive time T such that Hartree equation (1.1) has a unique solution $u \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-\alpha})$. Moreover, if T^* is the maximal existence time and is finite, then $\lim_{t \nearrow T^*} \|u(t)\|_{H^{\frac{\gamma}{2}}} = \infty$.*

Proof. The following is rather standard. So we shall be brief.

Let $(X(T, \rho), d)$ be a complete metric space with metric d defined by

$$X(T, \rho) = \{u \in L_T^\infty(H^s(\mathbb{R}^n)) : \|u\|_{L_T^\infty H^s} \leq \rho\}, \quad d_X(u, v) = \|u - v\|_{L_T^\infty L^2}.$$

We define a mapping $\mathcal{N} : u \mapsto \mathcal{N}(u)$ on $X(T, \rho)$ by

$$(4.1) \quad \mathcal{N}(u)(t) = U(t)\varphi - i \int_0^t U(t-t')F(u)(t') dt',$$

where $U(t) = e^{-it|\nabla|^\alpha}$. We use the standard contraction mapping argument. For $u \in X(T, \rho)$ and $s \geq \frac{\gamma}{2}$ we have

$$(4.2) \quad \begin{aligned} \|\mathcal{N}(u)\|_{L_T^\infty H^s} &\leq \|\varphi\|_{H^s} + T\|F(u)\|_{L_T^\infty H^s} \\ &\lesssim \|\varphi\|_{H^s} + T\left(\|V_\gamma(|u|^2)\|_{L_T^\infty L^\infty} \|u\|_{L_T^\infty H^s} \right. \\ &\quad \left. + \|V_\gamma(|u|^2)\|_{L_T^\infty H^{\frac{s}{2}}} \|u\|_{L_T^\infty L^{\frac{2n}{n-\gamma}}}\right) \\ &\lesssim \|\varphi\|_{H^s} + T\left(\|u\|_{L_T^\infty H^{\frac{\gamma}{2}}}^2 \|u\|_{L_T^\infty H^s} + \|u\|_{L_T^\infty L^{\frac{2n}{n-\gamma}}}^2 \|u\|_{L_T^\infty H^s}\right) \\ &\lesssim \|\varphi\|_{H^s} + T\|u\|_{L_T^\infty H^{\frac{\gamma}{2}}}^2 \|u\|_{L_T^\infty H^s} \lesssim \|\varphi\|_{H^s} + T\rho^3. \end{aligned}$$

Here we have used the trivial inequality

$$V_\gamma = \int_{\mathbb{R}^n} \frac{\psi(x-y)}{|x-y|^\gamma} |u(y)|^2 dy \leq \|\psi\|_{L^\infty} \int_{\mathbb{R}^n} |x-y|^{-\gamma} |u(y)|^2 dy,$$

generalized Leibniz rule, the Hardy-Sobolev inequality

$$(4.3) \quad \sup_{x \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} \frac{|u(x-y)|^2}{|y|^\gamma} dy \right| \lesssim \|u\|_{\dot{H}^{\frac{\gamma}{2}}}^2,$$

and we used the Sobolev embedding $H^{\frac{\gamma}{2}} \hookrightarrow L^{\frac{2n}{n-\gamma}}$. If we choose ρ and T such as $\|\varphi\|_{H^s} \leq \rho/2$ and $CT\rho^3 \leq \rho/2$, then \mathcal{N} maps $X(T, \rho)$ to itself.

Now we show that \mathcal{N} is a Lipschitz map for sufficiently small T . Let $u, v \in X(T, \rho)$. Then we have

$$\begin{aligned}
d_X(\mathcal{N}(u), \mathcal{N}(v)) &\lesssim T \|V_\gamma(|u|^2)u - V_\gamma(|v|^2)v\|_{L_T^\infty L^2} \\
&\lesssim T \left(\|V_\gamma(|u|^2)(u - v)\|_{L_T^\infty L^2} + \|V_\gamma(|u|^2 - |v|^2)v\|_{L_T^\infty L^2} \right) \\
&\lesssim T \left(\|u\|_{L_T^\infty H^{\frac{\gamma}{2}}}^2 d_X(u, v) + \|V_\gamma(|u|^2 - |v|^2)\|_{L_T^\infty L^{\frac{2n}{\gamma}}} \|v\|_{L_T^\infty L^{\frac{2n}{n-\gamma}}} \right) \\
&\lesssim T(\rho^2 d_X(u, v) + \rho \| |u|^2 - |v|^2 \|_{L_T^\infty L^{\frac{2n}{2n-\gamma}}}) \\
&\lesssim T(\rho^2 + \rho(\|u\|_{L_T^\infty L^{\frac{2n}{n-\gamma}}} + \|v\|_{L_T^\infty L^{\frac{2n}{n-\gamma}}})) d_X(u, v) \\
&\lesssim T\rho^2 d_X(u, v).
\end{aligned}$$

The above estimate implies that the mapping \mathcal{N} is a contraction, if T is sufficiently small. The uniqueness and time regularity follows easily from the equation (1.1) and a similar contraction argument.

Now let T^* be the maximal existence time. If $T^* < \infty$, then it is obvious from the estimate (4.2) and the standard local well-posedness theory that $\lim_{t \nearrow T^*} \|u(t)\|_{H^{\frac{\gamma}{2}}} = \infty$. This completes the proof of Proposition 4.1. \square

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